One-step Methods

Second-Order Runge-Kutta Method

Second-Order Runge-Kutta Method

$$y_{i+1} = y_i + \left(a_1 k_1 + a_2 k_2\right) h \qquad \text{-1}$$

where

$$k_1 = f\left(x_i, y_i\right) \qquad \text{-2}$$

$$k_2 = f\left(x_{i+p_1 h}, y_{i+q_{11} k_1 h}\right) \qquad \text{-3}$$

$a_1 = ?$ $a_2 = ?$ $p_1 = ?$ $q_{11} = ?$

Second-Order Runge-Kutta Method

Second-order Taylor series approximation

$$y_{i+1} = y_i + f\left(x_i, y_i\right)h + f'\left(x_i, y_i\right)\frac{h^2}{2} + O\left(h^2\right)$$

or

$$y_{i+1} = y_i + f\left(x_i, y_i\right)h + \left[\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\bullet\frac{dy}{dx}\right]\frac{h^2}{2}$$

-I

Second-Order Runge-Kutta Method

$$d\left[f\left(x_i, y_i\right)\right] = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$$

$$\frac{d}{dx}\left[f\left(x_i, y_i\right)\right] = \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx}$$

$$\frac{d}{dx}\left[f\left(x_i, y_i\right)\right] = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx}$$

Second-Order Runge-Kutta Method

Taylor series approximation for a two-variable function

$$g\left(x_{i+r}, y_{i+s}\right) = g\left(x_i, y_i\right)h + r\frac{\partial g}{\partial x} + s\frac{\partial g}{\partial y} + O\left(h^2\right)$$

or

$$f\left(x_{i+p_1 h}, y_{i+q_{11}k_1 h}\right) = f\left(x_i, y_i\right) + p_1 h\frac{\partial f}{\partial x} + q_{11}k_1\frac{\partial f}{\partial y} + O(h^2) \quad \text{-4}$$

Second-Order Runge-Kutta Method

Eq.4 to eq.3 and eq.3 to eq.2

$$k_2 = f\left(x_i, y_i\right) + p_1 h \frac{\partial f}{\partial x} + q_{11} k_1 h \frac{\partial f}{\partial y}$$

$$y_{i+1} = y_i + h\left\{a_1 \cdot f\left(x_i, y_i\right) + a_2 \cdot f\left(x_i, y_i\right) + p_1 h \frac{\partial f}{\partial x} + q_{11} k_1 h \frac{\partial f}{\partial y}\right\}$$

$$y_{i+1} = y_i + \left(a_1 + a_2\right) \cdot h \cdot f\left(x_i, y_i\right) + h^2 \left[a_1 p_1 \frac{\partial f}{\partial x} + a_2 q_{11} k_1 \frac{\partial f}{\partial y}\right]$$ -II

Second-Order Runge-Kutta Method

Eq.I = Eq.II

$$a_1 + a_2 = 1$$

$$a_2 p_1 = \frac{1}{2}$$

$$a_2 q_{11} = \frac{1}{2}$$

Optimal Second-Order RK Method

$$a_1 = \frac{1}{4}$$

$$a_2 = \frac{3}{4}$$

$$p_1 = \frac{2}{3}$$

$$q_{11} = \frac{2}{3}$$

Optimal Second-Order RK Method

$$y_{i+1} = y_i + \left(\frac{1}{4} k_1 + \frac{3}{4} k_2 \right) h$$

$$k_1 = f\left(x_i, y_i \right)$$

$$k_2 = f\left(x_{i+\frac{2}{3}h}, y_{i+\frac{2}{3}k_1 h} \right)$$

Second-Order RK : Heun method

$$a_1 = \frac{1}{2} \quad \Rightarrow \quad a_2 = \frac{1}{2}$$

$$p_1 = 1 \qquad q_{11} = 1$$

Second-Order RK : Heun method

$$y_{i+1} = y_i + \left(\frac{1}{2} k_1 + \frac{1}{2} k_2 \right) h$$

$$k_1 = f\left(x_i, y_i\right)$$

$$k_2 = f\left(x_{i+h}, y_{i+k_1 h}\right)$$

Second-Order RK : Ralston method

$$a_1 = \frac{1}{3} \quad \Rightarrow \quad a_2 = \frac{2}{3}$$

$$p_1 = \frac{3}{4} \qquad q_{11} = \frac{3}{4}$$

Second-Order RK : Ralston method

$$y_{i+1} = y_i + \left(\frac{1}{3} k_1 + \frac{2}{3} k_2 \right) h$$

$$k_1 = f\left(x_i, y_i \right)$$

$$k_2 = f\left(x_{i+\frac{3}{4}h}, y_{i+\frac{3}{4}k_1 h} \right)$$

Second-Order RK : Midpoint method

$$a_1 = 0 \quad \Rightarrow \quad a_2 = 1$$

$$p_1 = \frac{1}{2} \qquad q_{11} = \frac{1}{2}$$

Second-Order RK : Midpoint method

$$y_{i+1} = y_i + \left(k_2\right) h$$

$$k_1 = f\left(x_i, y_i\right)$$

$$k_2 = f\left(x_{i+\frac{1}{2}h}, y_{i+\frac{1}{2}k_1 h}\right)$$

Second-Order RK: General Form

$$y_{i+1} = y_i + \left(a_1 k_1 + a_2 k_2\right) h$$

where $k_1 = f\left(x_i, y_i\right)$ $k_2 = f\left(x_{i+p_1 h}, y_{i+q_{11} k_1 h}\right)$

Method	a_1	a_2	p_1	q_{11}
Midpoint	0	1	1/2	1/2
Ralston	1/3	2/3	3/4	3/4
Huen	1/2	1/2	1	1
Optimal	1/4	3/4	2/3	2/3

Third-Order RK: General Form

$$y_{i+1} = y_i + h(a_1 k_1 + a_2 k_2 + a_3 k_3)$$

where $k_1 = f(x_i, y_i)$ $k_2 = f\left(x_{i+hp_1}, y_{i+hk_1 q_{11}}\right)$

$$k_3 = f\left(x_{i+hp_2}, y_{i+h(k_1 q_{21} + k_2 q_{22})}\right)$$

Method	a_1	a_2	a_3	q_{11}	q_{21}	q_{22}	p_1	p_2
Classic	1/6	2/3	1/6	1/2	-1	2	1/2	1
Huen	1/4	0	3/4	1/3	0	2/3	1/3	2/3
Nystrom	2/8	3/8	3/8	2/3	0	2/3	2/3	2/3
Nearly optimal	2/9	3/9	4/9	1/2	0	3/4	1/2	3/4

Fourth-Order RK: General Form

$$y_{i+1} = y_i + h(a_1 k_1 + a_2 k_2 + a_3 k_3 + a_4 k_4)$$

where

$$k_1 = f(x_i, y_i) \qquad k_2 = f\left(x_{i+hp_1}, y_{i+hk_1 q_{11}}\right)$$

$$k_3 = f\left(x_{i+hp_2}, y_{i+h(k_1 q_{21} + k_2 q_{22})}\right)$$

$$k_4 = f\left(x_{i+hp_3}, y_{i+h(k_1 q_{31} + k_2 q_{32} + k_3 q_{33})}\right)$$

Method	a_1	a_2	a_3	a_4	q_{11}	q_{21}	q_{22}	q_{31}	q_{32}	q_{33}	p_1	p_2	p_3
Classic	1/6	1/3	1/3	1/6	1/2	0	1/2	0	0	1	1/2	1/2	1
Kutta	1/8	3/8	3/8	1/8	1/3	-1/3	1	1	-1	1	1/3	2/3	1

Runge-Kutta Method : General Form

$$y_{i+1} = y_i + h(a_1 k_1 + a_2 k_2 + \cdots + a_n k_n)$$

where

$$k_1 = f(x_i, y_i)$$

$$k_2 = f\left(x_{i+hp_1}, y_{i+hk_1 q_{11}}\right)$$

$$k_3 = f\left(x_{i+hp_2}, y_{i+h(k_1 q_{21} + k_2 q_{22})}\right)$$

$$k_4 = f\left(x_{i+hp_3}, y_{i+h(k_1 q_{31} + k_2 q_{32} + k_3 q_{33})}\right)$$

$$\vdots$$

$$k_n = f\left(x_{i+hp_{n-1}}, y_{i+h(k_1 q_{n-1,1} + \cdots + k_{n-1} q_{n-1,n-1})}\right)$$